6 INSTANTONS

$\mathbb{R}^4$ EUCLIDEAN YANG_MILLS THEORY
(PHYSICAL INTERPRETATION: PARTICLES IN 4+1 THEORY OR TUNNELING EVENTS IN EUCLIDEAN FORMULATION OF 3+1 THEORY)

$A_\mu \in su(2)$ (OR $SU(N)$)

YANG_MILLS THEORY IN 3+1 $(+,-,-,-)$

$$S = -\int d^4x \, \frac{1}{2} \, tr\left(F_{\mu\nu} F^{\mu\nu}\right)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right]$$

$$S = \int d^4x \left(tr\left(\vec{E}^2\right) - tr\left(\vec{B}^2\right)\right)$$

(EUCLIDEAN ROTATION $S = \int dt \left(\partial_t \phi^2 - V\right)$, $e^{iS} = e^{-S_E}$, $it = \tau$, $iS = -\int d\tau \left(\partial_\tau \phi^2 + V\right)$)

$t, x, y, z \longrightarrow it, x, y, z \qquad A_0, A_1, A_2, A_3 \longrightarrow -iA_0, A_1, A_2, A_3$

$$S_E = \int d^4x \, \frac{1}{2} \, tr\left(F^E_{\mu\nu} F^E_{\mu\nu}\right)$$

$A \to gA \qquad S \to \frac{1}{g^2} S$ FROM NOW ON WE RESCALE THE COUPLING SO THAT $D_\mu = \partial_\mu - i[A, \,]$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i\left[A_\mu, A_\nu\right]$.

FROM NOW ON WE REMOVE "E"

$$S = \frac{1}{2g^2}\int d^4x \; \mathrm{tr}\left(F_{\mu\nu}F_{\mu\nu}\right)$$

Euclidean Action

METRIC $(+,+,+,+)$

δ_μ^ν RAISE & LOWER INDICES

$$*F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}$$

$$\begin{cases} D_\mu F_{\mu\nu} = 0 & \text{Equation of Motion} \\ D_\mu * F_{\mu\nu} = 0 & \text{Bianchi Equation} \end{cases}$$

WE CAN SHOW THAT

$$\mathrm{tr}\left(F_{\mu\nu}^2\right) = \mathrm{tr}\left(*F_{\mu\nu}^2\right)$$

$$\mathrm{tr}\left(\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F_{\alpha\beta}\,\frac{1}{2}\epsilon_{\mu\nu\gamma\tau}F_{\gamma\tau}\right) =$$

$$= \frac{1}{4}\left(\delta_{\alpha\gamma}\delta_{\beta\tau} - \delta_{\alpha\tau}\delta_{\beta\gamma}\right)2\,\mathrm{tr}\left(F_{\alpha\beta}F_{\gamma\tau}\right)$$

$$= \mathrm{tr}\left(F_{\alpha\beta}^2\right)$$

WE WRITE THE EUCLIDEAN ACTION USING THE BOGO. TRICK

$$S = \frac{1}{4g^2}\int d^4x \left\{ \mathrm{tr}\left(F_{\mu\nu} \mp *F_{\mu\nu}\right)^2 \pm 2\,\mathrm{tr}\left(F_{\mu\nu} * F_{\mu\nu}\right)\right\}$$

WE USED THE PREVIOUS IDENTITY

$$F_{\mu\nu} = \pm * F_{\mu\nu}$$

BOGO. EQUATION (SELF-DUAL OR ANTI SELF-DUAL)

$$\pm \frac{1}{2g^2}\int d^4x\ \mathrm{tr}\left(F_{\mu\nu} * F_{\mu\nu}\right)$$ IS A TOPOLOGICAL TERM.

BACK TO THE GENERAL THEORY FOR A MOMENT.

$U(1)$ GAUGE ON $\mathbb{R}^2$, $C_1 = \frac{1}{2\pi}\int f_{12}\, d^2x$ IS THE "FIRST CHERN NUMBER". CHERN NUMBERS ARE TOPOLOGICAL INVARIANTS FOR GAUGE FIELDS IN EVEN DIMENSIONS.

$SO(2)$ GAUGE ON $\mathbb{R}^4$, WE HAVE THE "SECOND CHERN NUMBER":

$$C_2 = \frac{1}{16\pi^2}\int d^4x\ \mathrm{tr}\left(F_{\mu\nu} * F_{\mu\nu}\right)$$

FIRST WE HAVE TO SHOW THAT IS A TOTAL DERIVATIVE

$$C_2 = \int d^4x\ \partial_\mu J_\mu$$

$$J_\mu = \frac{1}{16\pi^2}\epsilon_{\mu\nu\rho\sigma}\,\mathrm{tr}\left(A_\nu F_{\rho\sigma} + i\,\frac{2}{3}A_\rho A_\sigma A_\nu\right)$$

$$\mathrm{tr}\left(F_{\mu\nu} * F_{\mu\nu}\right) =$$

$$= \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\,\mathrm{tr}\left(F_{\mu\nu} F_{\rho\sigma}\right) =$$

$$= 2\epsilon_{\mu\nu\rho\sigma} \operatorname{tr}\left((\partial_\mu A_\nu - i A_\mu A_\nu)(\partial_\rho A_\sigma - i A_\rho A_\sigma) \right)$$

$$= 2\epsilon_{\mu\nu\rho\sigma} \operatorname{tr}\left(\partial_\mu (A_\nu \partial_\rho A_\sigma) - 2i\, \partial_\mu A_\nu A_\rho A_\sigma \right)$$

$$= 2\partial_\mu \epsilon_{\mu\nu\rho\sigma} \operatorname{tr}\left(A_\nu \partial_\rho A_\sigma - \frac{2}{3} i A_\nu A_\rho A_\sigma \right)$$

$$= \partial_\mu \epsilon_{\mu\nu\rho\sigma} \operatorname{tr}\left(A_\nu F_{\rho\sigma} + i\frac{2}{3} A_\nu A_\rho A_\sigma \right)$$

SO WE HAVE

$$C_2 = \frac{1}{16\pi^2} \int d^4x \operatorname{tr}(F_{\mu\nu} * F_{\mu\nu})$$

$$= \int d^4x\, \partial_\mu \Big(\underbrace{\frac{1}{16\pi^2} \epsilon_{\mu\nu\rho\sigma} \operatorname{tr}\left(A_\nu F_{\rho\sigma} + i\frac{2}{3} A_\nu A_\rho A_\sigma \right)}_{J_\mu} \Big)$$

SO C_2 IS A BOUNDARY TERM.

BOGO. BOUND IS

$$\left| \frac{1}{2g^2} \int d^4x \operatorname{tr}(F_{\mu\nu} * F_{\mu\nu}) \right| = \frac{8\pi^2}{g^2} |C_2|$$

REMEMBER THAT A GAUGE TRANSFORMATION IS A UNITARY MATRIX $U(x) \in SU(2)$

$$A_\mu \to U A_\mu U^\dagger + i U \partial_\mu U^\dagger \qquad F_{\mu\nu} \to U F_{\mu\nu} U^\dagger$$

A "PURE" GAUGE FIELD HAS ZERO CURVATURE $F_{\mu\nu} = 0$, SO $A_\mu = +i U \partial_\mu U^\dagger$

FINITE ACTION CONFIGURATION MUST THEN

$$A_\mu \xrightarrow[x\to\infty]{} i U \partial_\mu U^\dagger$$ AT THE S^3 SPHERE AT INFINITY

U IS THUS A MAP

$$U: S^3 \longrightarrow S^3$$

∞ SPACE — SU(2) MANIFOLD

THUS CLASSIFYED BY THE TOPOLOGICAL NUMBER $\pi_3(SU(2)) = \pi_3(S^3) = \mathbb{Z}$.

THE CHERN CURRENT AT INFINITY IS:

$$J_\mu\big|_\infty = \frac{1}{16\pi^2}\epsilon_{\mu\nu\rho\sigma}\, tr\left(A_\nu \underbrace{F_{\rho\sigma}}_{0} + i\frac{2}{3}A_\nu A_\rho A_\sigma\right)$$

$$= \frac{1}{24\pi^2}\epsilon_{\mu\nu\rho\sigma}\, tr\left(U\partial_\nu U^\dagger U\partial_\rho U^\dagger U\partial_\sigma U^\dagger\right)$$

THIS IS IN FACT RELATED TO THE TOPOLOGICAL DEGREE OF THE MAP $U: S^3 \to S^3$.

CONSIDER O(4) SIGMA MODEL $\phi_1, \phi_2, \phi_3, \phi_4$

$\phi^a\phi^a = 1$ — S^3 MANIFOLD.

$$N = \int d^3x\, \frac{1}{V_{S^3}}\frac{1}{3!}\epsilon_{ijk}\epsilon_{abcd}\,\phi^a\partial_i\phi^b\partial_j\phi^c\partial_k\phi^d$$ $V_{S^3} = 2\pi^2$

$A_\mu = A_\mu^a t^a$ WITH NORMALIZATION

$$\mathrm{tr}(t^a t^b) = \frac{1}{2}\delta^{ab}$$

$$[t^a, t^b] = i\epsilon^{abc} t^c$$

$$S = \frac{1}{4g^2}\int d^4x\, F^a_{\mu\nu} F^a_{\mu\nu}$$

$$C_2 = \frac{1}{32\pi^2}\int d^4x\, F^a_{\mu\nu} * F^a_{\mu\nu}$$

$$J_\mu = \frac{1}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}\left(A^a_\nu F^a_{\rho\sigma} - \frac{1}{3}\epsilon^{abc} A^a_\nu A^b_\rho A^c_\sigma\right)$$

$$\left(J_\mu = \frac{1}{16\pi^2}\epsilon_{\mu\nu\rho\sigma}\left(A^a_\nu \partial_\rho A^a_\sigma + \frac{1}{3}\epsilon^{abc} A^a_\nu A^b_\rho A^c_\sigma\right)\right)$$

WE WILL USE BOTH FORMULATIONS, A_μ AND A_μ^a.

LET'S START WITH THE WINDING NUMBER 1.
THIS IS THE BPST INSTANTON SOLUTION.

WE INTRODUCE THE 't HOOFT SYMBOL

$$\eta_{a\mu\nu} = \begin{cases} \epsilon_{a\mu\nu} & \mu,\nu = 1,2,3 \\ \delta_{a\mu} & \nu = 4 \\ -\delta_{a\nu} & \mu = 4 \end{cases} \qquad \left(\begin{matrix} - \\ + \end{matrix}\ \text{IS FOR } \bar{\eta}_{a\mu\nu}\right)$$

$t_{\mu\nu} = \eta_{a\mu\nu} t_a$

$t_{ij} = \eta_{aij} t_a = \epsilon_{aij} t_a$

$t_{i4} = \eta_{ai4} t_a = t_i = -t_{4i}$

$t_{\mu\nu}$ IS SELF-DUAL

$$* t_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} t_{\rho\sigma} = t_{\mu\nu}$$

$$t_{\mu\nu} = \begin{pmatrix} 0 & t_3 & -t_2 & t_1 \\ -t_3 & 0 & t_1 & t_2 \\ t_2 & -t_1 & 0 & t_3 \\ -t_1 & -t_2 & -t_3 & 0 \end{pmatrix} \qquad \begin{array}{l} t_{12} = t_3 \\ * t_{12} = \frac{1}{2} \epsilon_{1234} t_{34} = t_3 \end{array}$$

ON THE CONTRARY

$\bar{t}_{\mu\nu} = \bar{\eta}_{a\mu\nu} t_a$ IS ANTI-SELF-DUAL

$$* \bar{t}_{\mu\nu} = -\bar{t}_{\mu\nu}$$

$t_{\mu\nu}$ & $\bar{t}_{\mu\nu}$ (AND $\eta_{a\mu\nu}$ & $\bar{\eta}_{a\mu\nu}$) ARE VERY USEFUL TO CONSTRUCT SOLUTIONS TO THE BPS EQUATIONS

$C_2 = +1$ $\quad S = \frac{8\pi}{g^2}$ IF THE FIELD IS SELF DUAL

$$F_{\mu\nu} = * F_{\mu\nu}$$

A GOOD ANSATZ IS

$$A_\mu = 2\sigma_{\mu\nu} x_\nu h(x^2)$$

AT $x \to \infty$, $A_\mu \to i U \partial_\mu U^\dagger$

$U = \hat{x}_4 \mathbb{1} + i\sigma_i \hat{x}_i$ UNIT TOPOLOGICAL MAP

$[1] \in \mathbb{Z} = \pi_3(S^3)$

$$A_\mu \to 2\sigma_{\mu\nu} x_\nu / x^2$$

SO $h(x^2) \longrightarrow \frac{1}{x^2}$

THE EXACT SOLUTION IS $h(x^2) = \frac{1}{x^2+\rho^2}$

$$\boxed{A_\mu = 2\sigma_{\mu\nu} \frac{x_\nu}{x^2+\rho^2}}$$

BPST INSTANTON WITH $C_2 = 1$

$$\boxed{A_\mu = 2\bar{\sigma}_{\mu\nu} \frac{x_\nu}{x^2+\rho^2}}$$

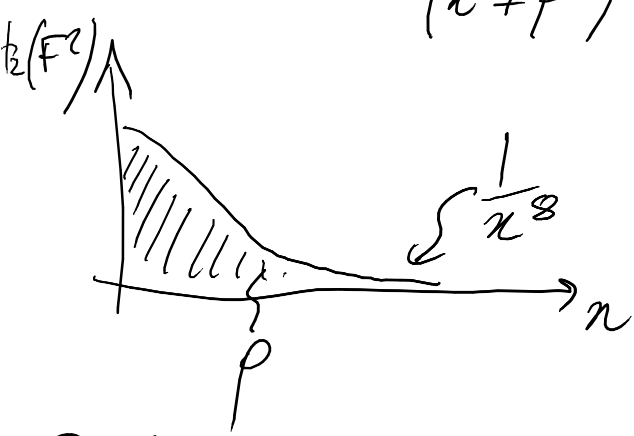
BPST ANTI-INSTANTON WITH $C_2 = -1$

$A_\mu \sim \frac{1}{r}$ AT LARGE DISTANCE, BUT THIS IS JUST A PURE GAUGE TAIL.

$F \sim \frac{1}{r^2}$ IS ZERO (OTHERWISE $\int r^3 dr \frac{1}{r^4}$ WOULD BE DIVERGENT)

THE LEADING TERM IS $F \sim \frac{1}{r^4}$

$$\mathrm{tr}\left(F_{\mu\nu}^2\right) = \frac{96\,\rho^4}{(x^2+\rho^2)^4} \quad \left(= \mathrm{tr}\left(\star F^2\right) = \mathrm{tr}\left(F \star F\right) \right)$$



ρ IS A FREE PARAMETER (ZERO MODE) DUE TO THE SPONTANEOUS BREAKING OF SCALE INVARIANCE (LIKE THE $\mathbb{CP}^1$ LUMP IN 2+1 DIMENSIONS)

IS SOMETIMES CONVENIENT TO USE THE SO CALLED "SINGULAR GAUGE".

$$A_\mu \to -\bar{A}_\mu + 2\,\bar{\sigma}_{\mu\nu}\frac{x_\nu}{x^2}$$ THIS IS A GAUGE TRANSF.

FOR THE 1 BPST INSTANTON

$$A_\mu = -2\,\bar{\sigma}_{\mu\nu}\,x_\nu\left(\frac{1}{x^2+\rho^2} - \frac{1}{x^2}\right)$$

$$C_2 = \underbrace{\frac{1}{16\pi^2}\int \mathrm{tr}\left(F_{\mu\nu} * F_{\mu\nu}\right) d^4x}_{\text{THIS IS GAUGE INVARIANT SO REMAINS THE SAME}} = \int_{\text{BOUNDARY}} ds^\mu\, J^\mu$$

THE BOUNDARY IS "0" NOW AND NOT "∞".

$$A_\mu = -\bar{\sigma}_{\mu\nu}\,\partial_\nu \log\psi$$

WITH $\psi = \dfrac{\rho^2}{x^2} + 1$ $\qquad \partial_\nu \log\psi = \dfrac{-2\rho^2 x_\nu}{x^2(x^2+\rho^2)}$

WE CAN THINK OF $A_\mu = -\bar{\sigma}_{\mu\nu}\,\partial_\nu \log\psi$ AS A GENERALIZED ANSANZ. THE SELF-DUAL EQUATION IN TERMS OF ψ BECOMES

$$F_{\mu\nu} = {}^*F_{\mu\nu} \Longrightarrow \partial_\mu\partial_\mu\psi = 0$$

THIS IS NOW A <u>LINEAR</u> EQUATION.

ϕ IS HARMONIC FUNCTION IN $\mathbb{R}^4$.

$\phi = \frac{\rho^2}{x^2} + 1$ IS ONE EXAMPLE THAT GIVES BPST [1] INSTANTON. MANY SOLUTIONS CAN NOW BE CONSTRUCTED WITH LINEAR SUPERPOSITION

$$\begin{cases} \phi = \sum_{j=1}^{N} \frac{\rho_j^2}{(x - a_j)^2} + 1 \\ A_\mu = -\bar{\sigma}_{\mu\nu} \partial_\nu \log \phi \end{cases}$$

(SAME FOR ANTI_ INSTANTONS)

"'t Hooft Ansatz"

$C_2 = N$

SO THIS IS A N-INSTANTON SOLUTION.

$\frac{1}{16\pi^2} \int \mathrm{tr}\left(F_{\mu\nu} * F_{\mu\nu}\right) d^4x$

REDUCES TO BOUNDARY INTEGRALS AROUND EACH SINGULARITY

FOR EACH POLE WE HAVE a_i^μ, ρ_i, 5 DEGREES OF FREEDOM

\# 't Hooft Solutions = $5N + 3$

3 GLOBAL SU(2) ROTATIONS

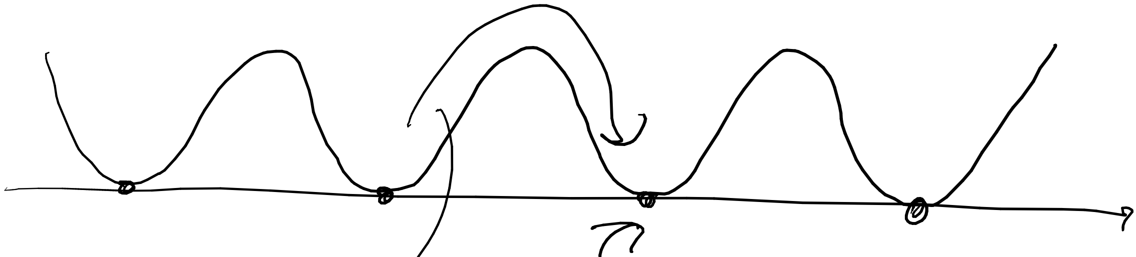
MODULI SPACE DIMENSIONS

$\mathbb{CP}^1$ LUMP IN 2+1	$4N$	2 POSITION, 1 SCALE, 1 PHASE FOR EACH
A.N.O. VORTEX IN 2+1	$2N$	2 POSITIONS
't H-P MONOPOLE IN 3+1	$4N$	3 POSITIONS + 1 GLOBAL U(1) ROTATION
INSTANTONS IN EUCLIDEAN	$8N$	4 POSITIONS, 1 SCALE, 3 SU(2) GLOBAL ROTATIONS

$\dim \mathcal{M}_N = 8N$ IS IN GENERAL MUCH BIGGER THAN $5N+3$ OBTAINED WITH 't Hooft ANSATZ. FULL SOLUTION REQUIRES MORE SOPHISTICATED TECHNIQUES.

ALL OF THESE HAVE GENERALIZATIONS IN WHICH THE GAUGE/GLOBAL SYMMETRY ARE BIGGER AND THE SOLITON MODULI SPACE BECOME RICHER.

INSTANTONS ARE TUNNELLING EVENTS BETWEEN DIFFERENT TOPOLOGICAL SECTORS



VACUUM $A_i = i\, U \partial_i U^+$

TUNNELING

$$P \propto e^{-S_I} = e^{-\frac{8\pi^2}{g^2}}$$

WE INDICATE WITH $|n\rangle$ THE STATES WITH $[n] \in \pi_3(S^3)$ UNITS. SINCE THERE IS TUNNELING, $|n\rangle$ ARE NOT THE REAL EIGENSTATES OF THE HAMILTONIAN.

H IS DIAGONALIZED ON THE "θ VACUA" BASIS:

$$|\theta\rangle = \sum_{n=-\infty}^{+\infty} e^{in\theta} |n\rangle$$

AN INSTANTON $|\theta\rangle \rightarrow e^{i\theta} |\theta\rangle$ LEAVES THE $|\theta\rangle$ STATE INVARIANT A PART FROM A PHASE MOLTIPLICATION.

IF WE ADD A TOPOLOGICAL TERM TO THE MINKOSKI ACTION

$$S_M = \int d^4x \left\{ -\frac{1}{2g^2} \mathrm{tr}\left(F_{\mu\nu} F_{\mu\nu}\right) + \alpha\, \mathrm{tr}\left(F_{\mu\nu} {}^*F_{\mu\nu}\right) \right\}$$

THIS IS EQUIVALENT TO CHOOSE A PARTICULAR $|\theta\rangle$ VACUUM.

$$e^{iS_M} = e^{i\alpha 16\pi^2 C_2} \implies \alpha = \frac{\theta}{16\pi^2}$$

$$\mathcal{L}_M = -\frac{1}{2g^2} \mathrm{tr}\left(F_{\mu\nu} F_{\mu\nu}\right) + \underbrace{\frac{\theta}{16\pi^2} \mathrm{tr}\left(F_{\mu\nu} {}^*F_{\mu\nu}\right)}$$

THIS DOES NOT CHANGE THE E_L CLASSICAL EQUATION OF MOTION.

BUT DUE TO THE INSTANTONS, IT HAS QUANTUM PHYSICAL EFFECTS.

7) DYONS

DYONS ARE OBJECTS WITH BOTH MAGNETIC AND ELECTRIC CHARGE. LET'S GO BACK TO THE GEORGI-GLASHOW THEORY

$$\mathcal{L} = -\frac{1}{2}\mathrm{tr}\left(F_{\mu\nu}^2\right) + \mathrm{tr}\left(D_\mu\phi\right)$$

WE CONSIDER THE BPS CASE $V(\phi)=0$

$$E_i = \partial_0 A_i - \partial_i A_0 - ig\left[A_0, A_i\right]$$

THERE ARE TWO CONVENIENT GAUGES. ONE IS THE STATIC GAUGE, $\partial_0(\ldots)=0$, THE OTHER IS THE $A_0=0$ GAUGE. WE USE THE STATIC ONE (JULIA-ZEE)

$$\vec{E} = -D_0\vec{A}$$

THE "ABELIAN" MAGNETIC FIELD WAS

$$f_{ij} = \sqrt{2}\,\frac{\mathrm{tr}\left(F_{ij}\phi\right)}{\sqrt{\mathrm{tr}\,\phi^2}} \qquad b_i = \frac{1}{2}\epsilon_{ijk} f_{jk} = \sqrt{2}\,\frac{\mathrm{tr}\left(B_i\phi\right)}{\sqrt{\mathrm{tr}\,\phi^2}}$$

THE "ABELIAN" ELECTRIC FIELD IS $e_i = \sqrt{2}\,\frac{\mathrm{tr}\left(E_i\phi\right)}{\sqrt{\mathrm{tr}\,\phi^2}}$

TO FIND A DYON IN THE STATIC GAUGE WE MUST ADD A_0 FIELD TO THE 'T HOOFT-POLYAKOV ANSATZ:

$$\begin{cases} \phi = v\,h(r)\,\hat{x}^a t^a \\ A_0 = J(r)\,\hat{x}^a t^a \\ A_i = \dfrac{1}{g}\left(1-K(r)\right)\epsilon_{ija}\dfrac{\hat{x}^j t^a}{r} \end{cases}$$

3D SCALARS

$J(r)$ IS AN EXTRA PROFILE FUNCTION

$$E_i = -\partial_i A_0 - ig\left[A_0, A_i\right]$$

$$= -t^a\left(J'\hat{x}^i\hat{x}^a + J\,\frac{\delta^{ia}-\hat{x}^i\hat{x}^a}{r}\right)$$

$$\underbrace{-iJ(1-K)\,\hat{x}^a\,\epsilon_{ijb}\frac{\hat{x}^j}{r}\left[t^a, t^b\right]}_{+J(1-K)\,\underbrace{\epsilon_{ijb}\epsilon_{abc}}_{\delta^{ic}\delta^{ja}-\delta^{ia}\delta^{cj}}\,\hat{x}^a\frac{\hat{x}^j}{r}t^c}$$

$$E_i = -t^a\left(J'\hat{x}^i\hat{x}^a + J\,\frac{\delta^{ia}-\hat{x}^i\hat{x}^a}{r}\right) + \frac{J(1-K)}{r}\left(t^i - \hat{x}^i\hat{x}^a t^a\right)$$

$$E_i = t^i\left(-J/r + \frac{J(1-K)}{r}\right) + t^a\hat{x}^a\hat{x}^i\left(-J' + \frac{J}{r} - \frac{J(1-K)}{r}\right)$$

$$E_i \xrightarrow[r\to\infty]{} -J' t^a \hat{x}^a \hat{x}^i \Rightarrow \text{ELECTRIC FIELD} \qquad J' \propto \frac{1}{r^2}$$

$$B_i \xrightarrow[r\to\infty]{} -\frac{1}{g}\frac{t^k\hat{x}^a\hat{x}^i}{r^2} \Rightarrow \text{MAGNETIC FIELD}$$

BOUNDARY CONDITION FOR J IS $J(0)=0$.
WE HAVE NO OTHER CONDITION.
SO WE WILL HAVE AN EXTRA PARAMETER OF SOLUTIONS.

IN THE BPS LIMIT WE CAN USE A NICE TRICK:

$$E = \int d^3x \left(\underbrace{\mathrm{tr}\left(\vec{E}^2\right)} + \mathrm{tr}\left(\vec{B}^2\right) + \mathrm{tr}\left(\vec{D}\phi^2\right) \right)$$

ONLY EXTRA TERM IN THE STATIC GAUGE

$$E = \int d^3x \left(\mathrm{tr}\left(\vec{E} + \sin\mu\, \vec{D}\phi\right)^2 + \mathrm{tr}\left(\vec{B} + \cos\mu\, \vec{D}\phi\right)^2 - 2\sin\mu\, \mathrm{tr}\left(\vec{E}\vec{D}\phi\right) - 2\cos\mu\, \mathrm{tr}\left(\vec{B}\vec{D}\phi\right) \right)$$

μ IS STILL TO BE FIXED.

BOGO. EQUATIONS

$$\begin{cases} \vec{E} + \sin\mu\, \vec{D}\phi = 0 \\ \vec{B} + \cos\mu\, \vec{D}\phi = 0 \end{cases}$$

$$\vec{E} = -\vec{D}A_0$$

THIS IS SOLVED BY $A_0 = \sin\mu\, \phi$

$$J(r) = \sin\mu\, v\, h(r)$$

$J \propto \frac{1}{r^2}$

$$E \geq \left| 2\sin\mu \underbrace{\int \mathrm{tr}(\vec{E}\vec{D}\phi)}_{\text{ELECTRIC CHARGE}} + 2\cos\mu \underbrace{\int \mathrm{tr}\,\vec{B}\vec{D}\phi}_{\text{MAGNETIC CHARGE}} \right|$$

$$E \geq \left| 2\sin\mu \int d\vec{S}\,\mathrm{tr}(\vec{E}\phi) + 2\cos\mu \int d\vec{S}\,\mathrm{tr}(\vec{B}\phi) \right|$$

$$b_i = \frac{q_m \hat{r}_i}{4\pi r^2}$$

$$E \geq v\left| \sin\mu\, q_e + \cos\mu\, q_m \right|$$

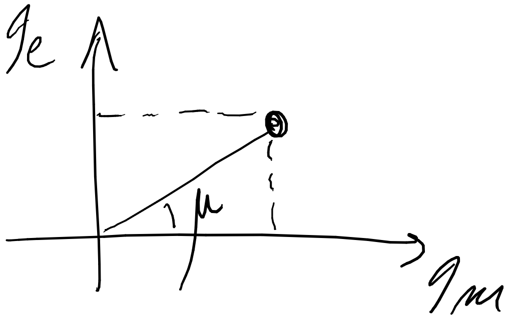
$$d_i = \frac{q_e}{4\pi r^2}\hat{r}_i$$

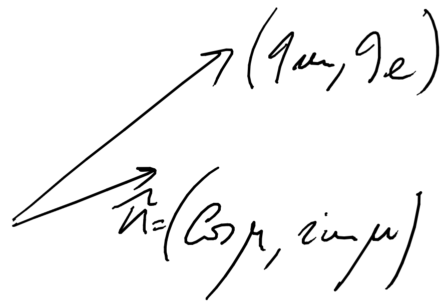
FOR MONOPOLE $q_m = -\frac{4\pi}{g}$, $q_e = 0$

TO MAXIMIZE THE BOUND WE NEED TO CHOOSE

$\mu = 0$ $\quad E \geq \frac{4\pi}{g} v$

IF WE HAVE A GENERIC q_m, q_e THE BOUND IS MAXIMIZED BY μ GIVEN BY





$\overline{(q_m, q_e)} \cdot \begin{pmatrix} \cos\mu \\ \sin\mu \end{pmatrix}$ IS MAXIMIZED WHEN THEY HAVE THE SAME ORIENTATION

$$E \geq v \sqrt{q_e^2 + q_m^2}$$

LENGTH OF THE (q_m, q_e) VECTOR

FOR THE SINGLE MONOPOLE

$$q_m = -\frac{4\pi}{g}$$

$$E_{BPS,\,DYON} = v \sqrt{q_e^2 + \frac{16\pi^2}{g^2}}$$

AT THE CLASSICAL LEVEL THE ELECTRIC CHARGE IS ARBITRARY.

SOLVING BPS EQUATIONS

$$\vec{E} + \sin\mu\, \vec{D}\phi = 0 \quad \Longrightarrow \quad f(r) = r \sin\mu\, h(r)$$

$$\vec{B} + \cos\mu\, \vec{D}\phi = 0$$

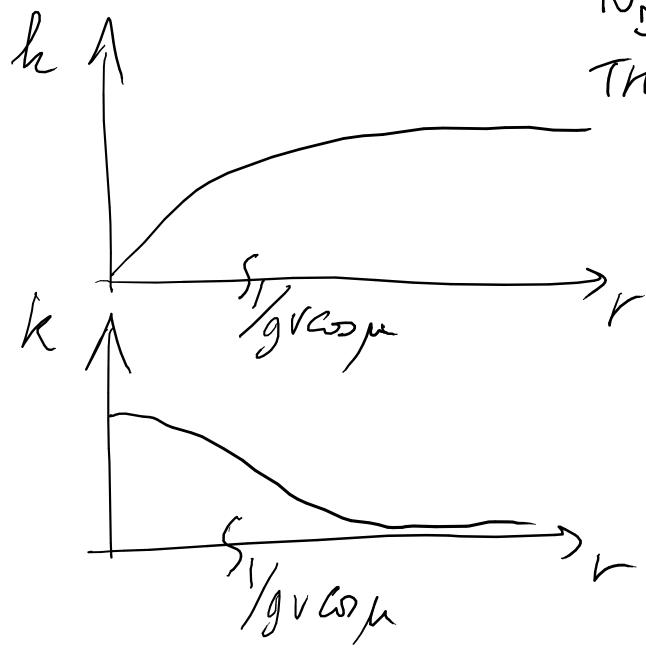
$$\Longrightarrow \begin{cases} \cos\mu\, v h k = -\dfrac{1}{g} k' \\ \cos\mu\, v h' = \dfrac{1-k^2}{r^2} \end{cases}$$

$r \longrightarrow g v \cos\mu\, r$ DIFFERENT RESCALING

SOLUTION IS

$$\begin{cases} h(r) = \coth(g v \cos\mu\, r) - \dfrac{1}{g v \cos\mu\, r} \\ k(r) = \dfrac{g v \cos\mu\, r}{\sinh(g v \cos\mu\, r)} \end{cases}$$

ELECTRIC FIELD REPULSION
R_D IS A BIT BIGGER
THAN R_M.



DIRAC QUANTIZATION CONDITION

$$q_e q_m = 2\pi n$$

FOR 1 MONOPOLE

$$q_e = \frac{g}{2}$$

CHAGE ELECTRON IN FUNDAMENTAL $SU(2)$

$$q_m = -\frac{4\pi}{g}$$

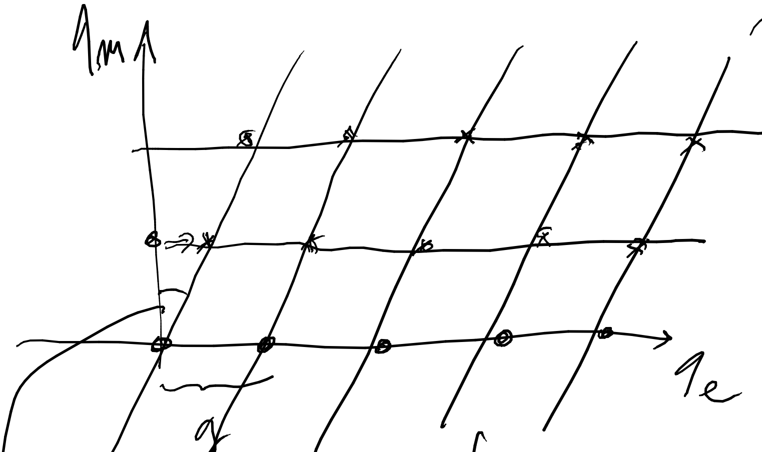
$n = -1$

GENERALIZATION IN CASE OF TWO DYONS

$$q_e^1, q_m^1 \qquad q_e^2, q_m^2$$

$$q_e^1 q_m^2 - q_e^2 q_m^1 = 2\pi n$$

$$\overbrace{q_e^1 \; q_m^1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} q_e^2 \\ q_m^2 \end{pmatrix} = 2\pi n$$



POSSIBLE DYON CHARGES

CHARGE W BOSON

θ ANGLE, WITTEN EFFECT.

$$M_D = v\sqrt{q_e + \frac{16\pi^2}{g^2}}$$

DYON CAN BE CONSIDERED AS A BOUND STATE MONOPOLE-W BOSONS